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Minimal harmonic graphs and their Lorentzian cousins

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ABSTRACT

Motivated by the observation that the only surface which is locally a graph of a harmonic function and is also a minimal surface in $\mathbb{E}^3$ is either a plane or a helicoid, we provide similar characterizations of the elliptic, hyperbolic and parabolic helicoids in $\mathbb{L}^3$ as the nontrivial zero mean curvature surfaces which also satisfy the harmonic equation, the wave equation, and a degenerate equation which is derived from the harmonic equation or the wave equation. This elementary and analytic result shows that the change of the roles of dependent and independent variables may be useful in solving differential equations.

1. Introduction

O. Kobayashi showed in 1983 that except the planes, helicoids are the only maximal surfaces in $\mathbb{L}^3 = (\mathbf{R}^3, dx^2 + dy^2 - dz^2)$ which are minimal with respect to the Riemannian metric $dx^2 + dy^2 + dz^2$ [3, Theorem 4.2].

His idea of proof is that any solution of the minimal and the maximal surface equation at the same time has straight lines as level curves. The analysis of the equation obtained by adding the minimal surface equation to the maximal surface equation plays a fundamental role in his proof.

We observe that his result holds when the maximal surface is replaced by timelike zero mean curvature surface and that by subtracting the minimal surface equation from the maximal surface equation one obtains the harmonic equation. Hence Kobayashi's result can be interpreted as follows:

Theorem 1.1. *(See Kobayashi [3].) Any two of the following conditions for $z = f(x, y)$ imply the remaining one and that the graph of $z = f(x, y)$ is either a plane or a helicoid, i.e. the scaled graph of $y = x \tan z$.*

(1) *f is a harmonic function.*
(2) *The graph of f in $\mathbb{E}^3$ is a minimal surface.*
(3) *The graph of f in $\mathbb{L}^3$ with z-axis as the time axis has the mean curvature zero.*

In $\mathbb{L}^3$, there are six different kinds of helicoids: spacelike or timelike depending upon the causal character of their normal vectors, and elliptic, hyperbolic or parabolic depending upon the causal character of their axes of the screw motion. The helicoid mentioned in the above theorem is the elliptic helicoid, and therefore, it would be interesting to know if other helicoids share similar properties.

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In this article, we provide an analytic proof of Kobayashi's result, and also prove similar results for other types of helicoids in $\mathbb{L}^3$. It turns out that whether the helicoid is spacelike or timelike does not matter, so we do not distinguish spacelike surfaces from timelike surfaces.

Theorem 1.2. *Let f be a C^2 function defined on a domain.*

(1) *If f is a solution of the harmonic equation $f_{xx} + f_{yy} = 0$ and if the graph of $t = f(x, y)$ is a zero mean curvature surface in $\mathbb{L}^3$, then the surface is a piece of either a plane or an elliptic helicoid in its standard position.*
(2) *If f is a solution of the wave equation $f_{tt} - f_{yy} = 0$ and if the graph of $x = f(y, t)$ is a zero mean curvature surface in $\mathbb{L}^3$, then the surface is a piece of either a plane or a hyperbolic helicoid in its standard position or the graph of $x = g(y + t)$ or $x = g(y - t)$ for an arbitrary twice differentiable function g.*
(3) *If f is a solution of the equation $f_{xx} - 2f_{xt} + f_{tt} = 0$ and if the graph of $y = f(x, t)$ is a zero mean curvature surface in $\mathbb{L}^3$, then the surface is a piece of either a plane or a parabolic helicoid in its standard position or a cylindrical zero mean curvature surface with a lightlike generator.*

The degenerate equation $f_{xx} - 2f_{xt} + f_{tt} = 0$ can be obtained as a limit of the harmonic equation or the wave equation as is explained at the end of Section 2. Hence one expects that the solutions of the limit equation are the limits of the solutions of the harmonic and/or the wave equations. But this is not necessarily true. In fact, if we look at the solution spaces of the systems of equations in Theorem 1.2, we find that the solutions to the harmonic equation have their rescaled limits, the parabolic helicoids, which are the solutions to the limit equation $f_{xx} - 2f_{xt} + f_{tt} = 0$. But the limit equation has one more family of solutions of lightlike cylinders $y = B(x - t)$ which does not come from a rescaled limit of solutions to the harmonic equation. Therefore it can be said that the solution spaces of our systems behave in an upper semi-continuous way at the parabolic case. It will be interesting if one can figure out whether this is a general phenomenon in the theory of differential equations.

2. Preliminaries

Consider

$$y = x\tan t, \qquad y = \frac{x-t}{x+t} - \frac{1}{6}(x+t)^2, \quad t = y\tanh x.$$

Their graphs in $\mathbb{L}^3$ are the union of a spacelike elliptic, parabolic, or hyperbolic helicoid and a timelike elliptic, parabolic, or hyperbolic helicoid, respectively. The spacelike helicoids and the timelike helicoids match analytically across their folded singularities, which is a general property of folded singularities of analytic surfaces with zero mean curvature [1,2].

A minimal graph $z = f(x, y)$ satisfy the minimal surface equation [4]

$$\bigl(1 + (f_y)^2\bigr) f_{xx} - 2 f_x f_y f_{xy} + \bigl(1 + (f_x)^2\bigr) f_{yy} = 0. \tag{2.1}$$

A zero mean curvature graph $t = f(x, y)$ in $\mathbb{L}^3$ satisfy the zero mean curvature equation

$$\bigl(1 - (f_y)^2\bigr) f_{xx} + 2 f_x f_y f_{xy} + \bigl(1 - (f_x)^2\bigr) f_{yy} = 0. \tag{2.2}$$

The graph is spacelike if $1 - (f_x^2 + f_y^2) > 0$, timelike if $1 - (f_x^2 + f_y^2) < 0$.

It is motivating that the Lorentzian metric

$$ds^2 = dx^2 + dy^2 - dt^2$$

can be viewed formally as the Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

if z is identified with it. This heuristic argument suggests that the wave equation $f_{tt} - f_{yy} = 0$ is an analogous equation in $\mathbb{L}^3$ to the harmonic equation $f_{xx} + f_{yy} = 0$ in $\mathbb{E}^3$. Therefore an analogous problem to Theorem 1.1 in $\mathbb{L}^3$ is to find the function $x = f(y, t)$ satisfying the wave equation $f_{tt} - f_{yy} = 0$ and whose graph is of zero mean curvature. This is the second part of Theorem 1.2.

Then the hyperbolic rotation together with scaling also gives an equation in the limit. In fact, we may apply, on the equations $f_{tt} - f_{yy} = 0$ or $f_{xx} + f_{yy} = 0$, the isometric coordinate changes by hyperbolic rotation around the y-axis

$$\begin{pmatrix} X \\ Y \\ T \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & -\sinh\theta \\ 0 & 1 & 0 \\ -\sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}.$$

Then we have

$$\partial_x = \cosh\theta\,\partial_X - \sinh\theta\,\partial_T,$$
$$\partial_y = \partial_Y,$$
$$\partial_t = -\sinh\theta\,\partial_X + \cosh\theta\,\partial_T.$$

And the operator $\partial_t^2 - \partial_y^2$ becomes

$$\partial_t^2 - \partial_y^2 = \sinh^2\theta\,\partial_X^2 - 2\sinh\theta\cosh\theta\,\partial_X\partial_T + \cosh^2\theta\,\partial_T^2 - \partial_Y^2.$$

As $\theta \to \infty$, the following scaled operator converges:

$$e^{-2\theta}\big(\partial_t^2 - \partial_y^2\big) \to \partial_X^2 - 2\partial_X\partial_T + \partial_T^2.$$

(We have similar computations for the operator $\partial_x^2 + \partial_y^2$.) Therefore the equation in the third part of Theorem 1.2 follows.

3. Proof of Theorem 1.2

Proof of the first part. Suppose $t = f(x, y)$ satisfies the zero mean curvature equation (2.2) and the harmonic equation:

$$f_{xx} + f_{yy} = 0. \tag{3.1}$$

To solve (2.1) and (3.1) for f, we view the graph of this function from the side, and think of this locally as a graph of the function $y = h(x,t)$. Now we need to find the appropriate equations for h.

First, since the graph of the function $y = h(x,t)$ is a zero mean curvature surface in its own, h satisfies

$$\big(-1 + (h_t)^2\big)h_{xx} - 2h_xh_th_{xt} + \big(1 + (h_x)^2\big)h_{tt} = 0. \tag{3.2}$$

On the other hand, the harmonic equation (3.1) becomes

$$(h_t)^2h_{xx} - 2h_xh_th_{xt} + \big(1 + (h_x)^2\big)h_{tt} = 0. \tag{3.3}$$

Now subtracting (3.3) from (3.2) we get

$$h_{xx} = 0, \tag{3.4}$$

and (3.3) becomes

$$\big(1 + (h_x)^2\big)h_{tt} - 2h_xh_th_{xt} = 0. \tag{3.5}$$

From Eq. (3.4) we get locally

$$h(x,t) = A(t)x + B(t)$$

for some functions $A(t)$ and $B(t)$ of t only. Plugging this into Eq. (3.5) we get

$$\big(\big(1 + A^2\big)A'' - 2A(A')^2\big)x + \big(1 + A^2\big)B'' - 2AA'B' = 0$$

and therefore

$$\big(1 + A^2\big)A'' - 2A(A')^2 = 0, \tag{3.6}$$
$$\big(1 + A^2\big)B'' - 2AA'B' = 0. \tag{3.7}$$

Suppose $A' \not\equiv 0$. Then, from (3.6) we obtain

$$A(t) = \tan(C_1t + C_2). \tag{3.8}$$

Now from (3.6) and (3.7) we get

$$\frac{A''}{A'} = \frac{B''}{B'} \quad \text{or} \quad B' \equiv 0$$

and hence

$$B(t) = C_3A(t) + C_4. \tag{3.9}$$

Combining (3.8) and (3.9) we get

$$h(x,t) = (x + C_3)\tan(C_1t + C_2) + C_4.$$

Under suitable parallel translations and homothety it is the graph of $y = x\tan t$. Therefore

$$t = f(x, y) = \tan^{-1}\left(\frac{y}{x}\right) \tag{3.10}$$

whose graph is a helicoid.

In case $A' \equiv 0$, A is constant, and from Eq. (3.7), $B'' \equiv 0$. Therefore h is a linear function and the graph is a plane.

Proof of the second part. We first note that $x = f(y, t)$ satisfies the following equations:

$$\big(1 + (f_y)^2\big) f_{tt} - 2 f_t f_y f_{ty} + \big(-1 + (f_t)^2\big) f_{yy} = 0, \tag{3.11}$$

$$f_{tt} - f_{yy} = 0. \tag{3.12}$$

To solve these two equations we view the graph of the function $x = f(y, t)$ as a graph of $t = h(x, y)$.

Now h satisfies

$$\big(1 - (h_x)^2\big)h_{yy} + 2h_x h_y h_{xy} + \big(1 - (h_y)^2\big)h_{xx} = 0. \tag{3.13}$$

On the other hand, the wave equation becomes

$$h_{xx}\big((h_y)^2 - 1\big) - 2h_{xy}h_x h_y + h_{yy}(h_x)^2 = 0. \tag{3.14}$$

Equating (3.14) with (3.13) we get the following two equations:

$$h_{yy} = 0, \tag{3.15}$$

$$h_{xx}\big(1 - (h_y)^2\big) + 2h_{xy}h_x h_y = 0. \tag{3.16}$$

From (3.15) we get locally

$$h(x, y) = A(x)y + B(x)$$

for some functions $A(x)$ and $B(x)$ of x only. Plugging this into (3.16) we get

$$\big(\big(1 - A^2\big)A'' + 2(A')^2 A\big)y + \big(1 - A^2\big)B'' + 2A'B'A = 0,$$

and therefore

$$\big(1 - A^2\big)A'' + 2(A')^2 A = 0, \tag{3.17}$$

$$\big(1 - A^2\big)B'' + 2A'B'A = 0. \tag{3.18}$$

Suppose $A' \not\equiv 0$. Then, from (3.17) we can write A in the form

$$A = \tanh(C_1 x + C_2). \tag{3.19}$$

Also from (3.17) and (3.18) we get

$$\frac{A''}{A'} = \frac{B''}{B'} \quad \text{or} \quad B' \equiv 0$$

and hence

$$B = C_3 A + C_4. \tag{3.20}$$

Combining (3.19) and (3.20) we get

$$h(x, y) = (y + C_3)\tanh(C_1 x + C_2) + C_4.$$

Under suitable scaling and homothety this becomes $h = y \tanh x$. Therefore

$$x = f(y, t) = \tanh^{-1}(t/y).$$

The surface given by this equation is the hyperbolic helicoid in $\mathbb{L}^3$.

In case $A' \equiv 0$ but $A^2 \not\equiv 1$, the surface is a plane.

In case $A^2 \equiv 1$, the surface is $h(x, y) = y + B(x)$ or $h(x, y) = -y + B(x)$. In terms of the original coordinates,

$$x = g(y + t) \quad \text{or} \quad x = g(y - t) \tag{3.21}$$

for a twice differentiable function g.

Proof of the third part. We let $y = f(x,t)$ be a C^2 function which satisfies the following equations:

$$f_{xx} - 2f_{xt} + f_{tt} = 0, \tag{3.22}$$

$$\left(1 + (f_x)^2\right)f_{tt} - 2f_x f_t f_{xt} + \left(-1 + (f_t)^2\right)f_{xx} = 0 \tag{3.23}$$

in $\mathbb{L}^3$. We change coordinates as follows: $u = x + t$, $v = x - t$. Under the coordinates (u, v), (3.23) becomes

$$f_u^2 f_{uv} - (1 + 2f_u f_v) f_{uv} + f_v^2 f_{uu} = 0. \tag{3.24}$$

Also from (3.22) we have

$$f_{vv} = 0, \tag{3.25}$$

from which $f = A(u)v + B(u)$. Plugging this into (3.24) we obtain

$$v\left(-2A(A')^2 + A^2A''\right) + \left(-A' - 2AA'B' + A^2B''\right) = 0, \tag{3.26}$$

and therefore

$$-2A(A')^2 + A^2A'' = 0, \tag{3.27}$$

$$-A' - 2AA'B' + A^2B'' = 0. \tag{3.28}$$

Suppose $A'(u) \not\equiv 0$. From (3.27), we obtain

$$\frac{A''}{A'} = \frac{2A'}{A},$$

and

$$A = \frac{1}{Cu + D}.$$

Under suitable translation of coordinates,

$$A = \frac{1}{Cu}.$$

Plugging this into (3.28), we have

$$uB'' + 2B' + Cu = 0. \tag{3.29}$$

This has the general solution of the form $B(u) = -\frac{C}{6}u^2 + \frac{C_1}{u} + C_2$. Hence

$$f(u, v) = \frac{v}{Cu} - \frac{C}{6}u^2 + \frac{C_1}{u} + C_2.$$

Under suitable translations in y and v directions and homothety, this function becomes $f(u, v) = \frac{v}{u} - \frac{1}{6}u^2$, hence

$$y = f(x,t) = \frac{x - t}{x + t} - \frac{1}{6}(x + t)^2,$$

whose graph is a parabolic helicoid.

Suppose $A'(u) \equiv 0$. In case $A(u) \equiv \text{constant} \neq 0$ the surface is a plane.

In case $A(u) \equiv 0$, $y = B(x + t)$ $(B \in C^2)$ which is a cylindrical timelike zero mean curvature surface [5]. This proves the theorem.

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